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A SIMPLE PRESENTATION FOR THE MAPPING CLASS GROUP OF AN ORIENTABLE SURFACE

BY
BRONISLAW WAJNRYB

ABSTRACT

Let $F_{n,k}$ be an orientable compact surface of genus n with k boundary components. For a suitable choice of $2n+1$ simple closed curves on $F_{n,1}$ the corresponding Dehn twists generate both $M_{n,0}$ and $M_{n,1}$. A complete system of relations is determined for these generators and the presentations of $M_{n,0}$ and $M_{n,1}$ obtained in this way are much simpler than the known presentations.

1. Introduction

Let $F_{n,k}$ be an orientable surface of genus n with k boundary components. The mapping class group $M_{n,k}$ of $F_{n,k}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms of $F_{n,k}$ which are the identity on the boundary. The goal of this paper is to find simple presentations for $M_{n,0}$ and $M_{n,1}$. These groups are generated by Dehn twists ([5]). Lickorish proved in [9] that $3n-1$ such twists are enough. Recently Humphries proved that $2n+1$ twists with respect to curves $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n, \delta$ (Fig. 1) generate $M_{n,0}$ and $M_{n,1}$ ([8]). This is the minimal possible number of twist generators. The first presentation for $M_{2,0}$ was obtained by Birman and Hilden in [3], who completed a program begun by Bergau and Mennicke in [1]. A presentation for the general case was obtained by Hatcher and Thurston in [7]. Their presentation is rather complicated and requires very many generators and relations. It was slightly simplified by Harer in [6]. Using their results we shall obtain a presentation with Humphries generators.

THEOREM 1. *The mapping class group $M_{n,1}$ admits a presentation with generators $a_1, b_1, \cdots, a_n, b_n, d$ and relations*

(A) $a_ib_ia_i = b_ia_ib_i$, $a_{i+1}b_ia_{i+1} = b_ia_{i+1}b_i$, $b_2db_2 = db_2d$, *every other pair of generators commutes.*

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(B) $(a_1b_1a_2)^4 = d(b_2a_2b_1a_1a_1b_1a_2b_2)^{-1}db_2a_2b_1a_1a_1b_1a_2b_2$.

(C) $dt_2dt_2^{-1}t_1t_2d(a_1a_2a_3t_1t_2)^{-1} = (ub_1a_2b_2a_3b_3)^{-1} \vee ub_1a_2b_2a_3b_3$ *where* $t_1 = b_1a_1a_2b_1$, $t_2 = b_2a_2a_3b_2$, $u = a_3b_3t_2d(a_3b_3t_2)^{-1}$, $v = a_1b_1a_2b_2d(a_1b_1a_2b_2)^{-1}$.

Elements a_i, b_i, d in Theorem 1 can be interpreted as Dehn twists with respect to curves α_i, β_i, δ (Fig. 1).

THEOREM 2. *The mapping class group* $M_{n,0}$ *admits a presentation with generators* $a_1, b_1, \cdots, a_n, b_n, d$ *and relations* (A), (B), (C) *and the following relation* (D).

(D) d_n *commutes with* $b_na_n \cdots b_1a_1a_1b_1 \cdots a_nb_n$ *where*

$$d_n = (u_1u_2 \cdots u_{n-1})^{-1}a_1u_1u_2 \cdots u_{n-1},$$

$$u_i = b_ia_{i+1}b_{i+1}v_i(b_{i+1}a_{i+1}b_ia_i)^{-1} \qquad \textit{for } i = 1, \cdots, n-1,$$

$$v_1 = d, \quad v_i = t_{i-1}t_iv_{i-1}(t_{i-1}t_i)^{-1} \qquad \textit{for } i = 2, \cdots, n-1,$$

$$t_i = b_ia_ia_{i+1}b_i \qquad \textit{for } i = 1, \cdots, n-1.$$

Element d_n in relation (D) represents a Dehn twist with respect to curve δ_n (Fig. 5).

REMARK. For $n = 1$ there is only one relation (A) in the presentations of the mapping class groups. For $n = 2$ we can omit relation (C). A Birman–Hilden presentation of $M_{2,0}$ contains relations (A), (D) and two more, $(a_1b_1a_2b_2d)^6 = 1$ and $(db_2a_2b_1a_1a_1b_1a_2b_2d)^2 = 1$, which can be replaced by relation (B). For $n > 2$ relation (C) does not follow from (A) and (B). If we write the relations in the form $R = 1$ then relations (A) and (D) have degree 0, relation (B) has degree 10, and relation (C) has degree 1.

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2. Notation and definitions

We shall often refer to Fig. 1 which represents a surface $F_{n,1} = F$. We shall not distinguish between isotopic simple closed curves on F, also called *circles*. If we

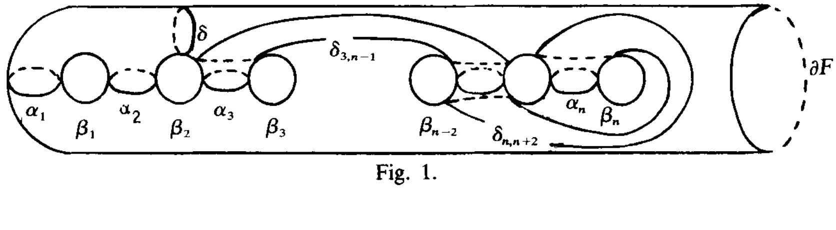

Fig. 1.

cut F along α_i (Fig. 1) we get a surface with two new boundary components, the upper component is γ_i and the lower component is γ_{2n+1-i}. We denote by δ_{ij}, $1 \leqq i < j \leqq 2n$, a circle on $F \backslash (\bigcup \alpha_i)$ which grabs circles γ_i and γ_j from the back and connects them in front (Fig. 1). Sometimes we call γ_i a hole in $F \backslash (\bigcup \alpha_i)$. When we say that a circle γ on F *contains* holes $\gamma_{i_1}, \cdots, \gamma_{i_k}$, we mean that γ does not meet $(\bigcup \alpha_i)$ and it separates $F \backslash (\bigcup \alpha_i)$ into two components, one of which contains exactly the holes $\gamma_{i_1}, \cdots, \gamma_{i_k}$ and does not contain the boundary ∂F. In particular δ_{ij} contains γ_i and γ_j.

Definition. Let γ be a circle on F and let N be a neighbourhood of γ homeomorphic to a standard annulus in $\mathbf{R}^2$ with its usual orientation. Let $h : F \to F$ be a homeomorphism, such that h is the identity outside N and inside N the concentric circles rotate counterclockwise while the rotation's angle increases from 0 to 2π going inwards. Any homeomorphism isotopic to h will be called a *Dehn Twist* along γ and will be denoted also by γ, and its inverse will be denoted by $\bar{\gamma}$.

We shall denote the mapping class group of F by G'. Composition of homeomorphisms in G' will be written from left to right. If $u \in G'$ and γ is a circle on F then the image of γ by u is denoted $(\gamma)u = \gamma'$. The Dehn twist along γ' equals $\bar{u}\gamma u$, where by $\bar{u}$ we denote the inverse of u in G'. Since we denote Dehn twist by the same letter as the corresponding circle we shall often write $(\gamma)u$ instead of $\bar{u}\gamma u$. More generally we shall write $(v)u = \bar{u}vu$ for any two elements $u, v \in G'$. If a bracket is separated from the next expression by a dot then it should be composed with the next expression in the usual way, $(v) \cdot u = vu$. The same notation will be used in groups G and M which appear in sections 4 and 5.

In the above notation a relation $aba = bab$ is equivalent to $(a)b = (b)\bar{a}$. In particular the relations in (A) can be written in the form $(a_i)b_i = (b_i)\bar{a}_i$, $(a_{i+1})b_i = (b_i)\bar{a}_{i+1}$.

3. A presentation of $M_{n,1}$ by Hatcher and Thurston

Hatcher and Thurston found a presentation of $M_{n,0}$ in [7]. They did not carry out their computations till the end. The main theorems in their paper are clearly true for any surface with boundary. In particular a presentation of $M_{n,1}$ can be obtained from their results. This was done by Harer who obtained an explicit presentation after some simplifications. We want to use his results with slightly different generators so we shall repeat part of the argument.

A cut system on F is an isotopy class of a collection $\{C_1,\cdots,C_n\}$ of disjoint circles on F, such that $F\setminus(\bigcup C_i)$ is a sphere with holes. If we replace some circle C_i by C'_i which intersects C_i transversally at one point and misses other C_k's, we get a new cut system which is obtained from the old one by a *simple move*, $(C_i)\to(C'_i)$. We suppress circles which don't change. Let X_1 be a graph with one vertex for each cut system on F and one edge between every pair of vertices connected by a simple move. X_2 is obtained from X_1 by attaching a 2-cell to each cycle of simple moves of one of the following types:

(L3) $(C_i)\to(C'_i)\to(C''_i)\to(C_i)$ (triangle)

(L4) $(C_i,C_j)\to(C'_i,C_j)\to(C'_i,C'_j)\to(C_i,C'_j)\to(C_i,C_j)$ (square)

(L5) $(C_i,C_j)\to(C_i,C'_j)\to(C'_i,C'_j)\to(C'_i,C''_j)\to(C_j,C''_j)\to(C_j,C_i)$ (pentagon)

where we assume that all the simple moves are possible. Then X_2 is connected and simply connected.

Let us fix the cut system $C=\{\alpha_1,\cdots,\alpha_n\}$. Let α_i, β_i for $i=1,\cdots,n$ and $\delta_{i,j}$ for $1\leqq i<j\leqq 2n$ be Dehn twists with respect to curves of the same name on Fig. 1. The mapping class group G' is generated by α_i's, δ_{ij}'s, $\sigma=\alpha_n\beta_n\alpha_n$, $\xi=\beta_n\alpha_n\alpha_n\beta_n$, and $\tau_i=\beta_i\alpha_i\alpha_{i+1}\beta_i$ for $i=1,\cdots,n-1$. Here τ_i permutes circles α_i and α_{i+1}, ξ reverses the orientation of α_n and σ permutes α_n and β_n (a simple move). We shall consider first certain subgroups of G' and their presentations (Harer).

H'_0 is the subgroup of elements of G' which leave circles α_i fixed. It is generated by α_i's and δ_{ij}'s and admits a presentation with relations

(i)a. α_i commutes with α_j for all i, j.

(i)b. α_k commutes with δ_{ij} for all i, j, k.

(i)c. $(\delta_{ij})\delta_{kl}=\delta_{ij}$ for $i<j<k<l$ or $i<k<l<j$,

$(\delta_{ij})\delta_{ik}=(\delta_{ij})\bar{\delta}_{jk}$ for $i<j<k$,

$(\delta_{ik})\delta_{ij}\delta_{ik}=(\delta_{ik})\bar{\delta}_{jk}$ for $i<j<k$,

$(\delta_{ik})\bar{\delta}_{ij}\bar{\delta}_{il}\delta_{ij}\delta_{il}=(\delta_{ik})\bar{\delta}_{jl}$ for $i<j<k<l$.

H' is the subgroup of elements of G' which leave the cut system $\{\alpha_1,\cdots,\alpha_n\}$ invariant. It is generated by H'_0 and ξ and $\tau_1,\cdots,\tau_{n-1}$. H' is defined by the following exact sequences:

$$1\to H'_0\to H'\xrightarrow{\theta}\pm\Sigma_n\to 1,\qquad 1\to(Z/2Z)^n\to\pm\Sigma_n\to\Sigma_n\to 1$$

where $\theta(\xi)\in(Z/2Z)^n$ and $\theta(\tau_i)$ is the transposition $(i,i+1)$ in the symmetric group Σ_n. Therefore H' is defined by relations (i) and

(ii)a. $(\tau_{i+1})\bar{\tau}_i=(\tau_i)\tau_{i+1}$, $(\tau_i)\tau_j=\tau_i$ for $|i-j|>1$,

(ii)b. $\tau_i^2\in H'_0$ for all i,

(ii)c. $\xi^2\in H'_0$,

(ii)d. $[(\xi)\tau_{n-1}\tau_{n-2}\cdots\tau_i, (\xi)\tau_{n-1}\tau_{n-2}\cdots\tau_j] \in H_0'$ for $i<j$,

(ii)e. $[\tau_i, (\xi)\tau_{n-1}\tau_{n-2}\cdots\tau_j] \in H_0'$ for $i \neq j$, $i \neq j-1$,

(iii)a. $(\alpha_i)\xi \in H_0'$,

(iii)b. $(\alpha_i)\tau_j \in H_0'$ for all i, j,

(iii)c. $(\delta_{ij})\xi \in H_0'$ for all i, j,

(iii)d. $(\delta_{ij})\tau_k \in H_0'$ for all i, j, k,

where the corresponding elements of H_0' can be explicitly computed.

The group G' is generated by H' and σ. All relations involving σ come from I and II below and from cycles of simple moves of type (L3), (L4), (L5). Hatcher and Thurston prove that the following relations suffice for the presentation of G' (together with (i)–(iii)).

I. σ commutes with $H'(\alpha_n, \beta_n)$ where $H'(\alpha_n, \beta_n)$ is the subgroup of the elements of H' which keep α_n and β_n invariant.

II. $\sigma^2 \in H'$.

III. $\sigma h \sigma h \sigma \in H'$ whenever there is a circle γ on F and a map $h \in H'$, such that γ intersects α_n and β_n at one point each and misses other α_i's and $(\gamma)\sigma h = \beta_n$, $(\beta_n)\sigma h = \alpha_n$, $(\alpha_n)\sigma h = \gamma$. All "triangle" relations follow from these.

IV. σ commutes with $h\sigma\bar{h}$ where $h \in H'$ takes the circle β on Fig. 4 onto β_n. All "square" relations follow from this.

V. $\sigma h_1 \sigma h_2 \sigma h_3 \sigma h_4 \sigma \in H'$ whenever there is a circle γ on F and $h_1, h_2, h_3, h_4 \in H'$, such that γ intersects α_{n-1} and β_n at one point each and misses β_{n-1} and other α_i's and $(\beta_{n-1})\sigma h_1 = \beta_n$, $(\gamma)\sigma h_1 \sigma h_2 = \beta_n$, $(\alpha_n)\sigma h_1 \sigma h_2 \sigma h_3 = \beta_n$, and $(\alpha_{n-1})\sigma h_1 \sigma h_2 \sigma h_3 \sigma h_4 = \beta_n$. All "pentagon" relations follow from these.

In order to write explicit relations we have to find generators for $H'(\alpha_n, \beta_n)$, and to find elements of H' corresponding to possible choices of γ's. Harer proves in [6] that in relations (III) four choices of γ suffice and we shall see that in (V) one choice of γ suffices.

The group $H'(\alpha_n, \beta_n)$ is generated by a subgroup $H_0'(\alpha_n, \beta_n)$ which leaves β_n and all α_i's fixed and by elements which permute α_i's for $i<n$ and reverse orientations of all α_i's. Maps τ_i, $i<n-1$, permute α_i's. σ^2 reverses the orientations of α_n and β_n. $(\sigma^2)\delta_{n-1,n}\bar{\tau}_{n-1}$ reverses the orientation of α_{n-1}. Let us cut F along β_n and all α_i's. We get one big hole $\alpha = \alpha_n \cup \beta_n$ and small holes γ_i, $1 \leqq i \leqq n-1$ or $n+2 \leqq i \leqq 2n$. $H_0'(\alpha_n, \beta_n)$ is generated by twists around holes and twists with respect to a standard set of loops which contain two holes each. σ^4 is a twist around α. α_{n-1} together with its conjugates by τ_i's provide other twists around holes, with suitable identifications. Loops not containing α are

obtained from $\delta_{n-2,n+2}$ and $\delta_{n-1,n+2}$ by conjugation by the above described elements of $H'(\alpha_n, \beta_n)$. Dehn twist around loop γ (Fig. 2), which contains holes

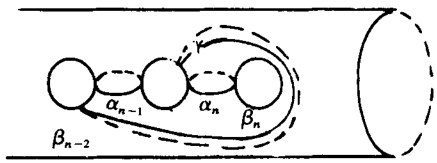

Fig. 2.

γ_{n+1} and α, equals $\delta_{n,n+1} \cdot \delta_{n,n+2} \cdot \delta_{n+1,n+2} \cdot \bar{\alpha}_{n-1} \cdot \bar{\alpha}_n^2$. Other standard loops containing α are obtained by conjugation. Therefore, $H'(\alpha_n, \beta_n)$ is generated by

$$\alpha_{n-1}, \tau_1, \cdots, \tau_{n-2}, \sigma^2, (\sigma^2)\delta_{n-1,n}\bar{\tau}_{n-1}, \delta_{n-2,n+2}, \delta_{n-1,n+2}$$

$$= (\sigma^4)\delta_{n-1,n}\bar{\tau}_{n-1}, \delta_{n,n+1} \cdot \delta_{n,n+2} \cdot \delta_{n+1,n+2} \cdot \bar{\alpha}_{n-1} \cdot \bar{\alpha}_n^2.$$

Therefore relation I produces

(iv) σ commutes with the following elements of H': $\alpha_{n-1}, \tau_1, \cdots, \tau_{n-2}, \delta_{n-2,n+2}$, $(\sigma^2)\delta_{n-1,n}\bar{\tau}_{n-1}$, $\delta_{n,n+1} \cdot \delta_{n,n+2} \cdot \delta_{n+1,n+2} \cdot \bar{\alpha}_{n-1} \cdot \bar{\alpha}_n^2$.

Relation II produces

(v) $\sigma^2 \in H'$.

LEMMA (Harer). *Let us cut F along $\alpha_1, \cdots, \alpha_{n-1}$. In the relation* III *it is enough to consider only such a loop γ on $F \setminus \{\alpha_1, \cdots, \alpha_{n-1}\}$ which starts at the intersection point of α_n and β_n, surrounds* 0, 1 *or* 2 *holes γ_i, comes back to the initial point, then goes once along β_n and once along α_n and comes back to the initial point. There are four cases up to the action of H'.*

By the lemma of Harer we have to consider only four cases, and for each case we have to find an element $h \in H'$ which satisfies the conditions in (III).

Case a. γ does not surround any hole (Fig. 3a). $h = \alpha_n$.

Case b. γ surrounds one hole γ_{n-1} (Fig. 3b). $h = \delta_{n-1,n}$.

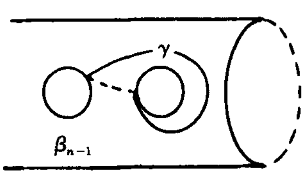

Fig. 3a.

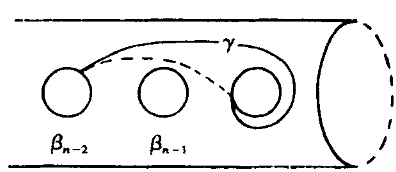

Fig. 3b.

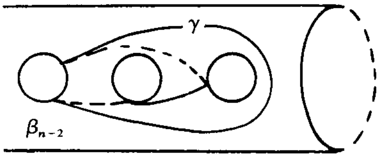

Fig. 3c.

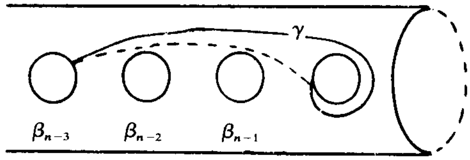

Fig. 3d.

Case c. γ surrounds two holes γ_{n-1} and γ_{n+2}, corresponding to the same circle α_{n-1} (Fig. 3c).

$$h = \delta_{n-1,n+2} \cdot \delta_{n,n+2} \cdot \delta_{n-1,n} \cdot \bar{\alpha}^2_{n-1} \cdot \bar{\alpha}_n .$$

Case d. γ surrounds two holes γ_{n-2} and γ_{n-1}, corresponding to different circles α_{n-2} and α_{n-1} (Fig. 3d).

$$h = \delta_{n-2,n-1} \cdot \delta_{n-2,n} \cdot \delta_{n-1,n} \cdot \bar{\alpha}_{n-2} \cdot \bar{\alpha}_{n-1} \cdot \bar{\alpha}_n .$$

Therefore the "triangle" relations produce

(vi) $\sigma h \sigma h \sigma \in H'$ for each of the following choices of $h : h = \alpha_n$, $h = \delta_{n-1,n}$,

$$h = \delta_{n-1,n+2} \cdot \delta_{n,n+2} \cdot \delta_{n-1,n} \cdot \bar{\alpha}^2_{n-1} \cdot \bar{\alpha}_n ,$$

$$h = \delta_{n-2,n-1} \cdot \delta_{n-2,n} \cdot \delta_{n-1,n} \cdot \bar{\alpha}_{n-2} \cdot \bar{\alpha}_{n-1} \cdot \bar{\alpha}_n .$$

In the relation (IV) we can choose $h = \bar{\delta}_{n-1,n} \cdot \tau_{n-1} \cdot \alpha_n$ and get

(vii) σ commutes with $h\sigma\bar{h}$ for $h = \bar{\delta}_{n-1,n} \cdot \tau_{n-1} \cdot \alpha_n$.

Finally we have relation (V) which corresponds to a cycle of simple moves of a form

$$(\alpha_{n-1}, \alpha_n) \to (\alpha_{n-1}, \beta_n) \to (\beta_{n-1}, \beta_n) \to (\beta_{n-1}, \gamma) \to (\alpha_n, \gamma) \to (\alpha_n, \alpha_{n-1}).$$

Consider curves β and γ' on Fig. 4. The curve β intersects γ and γ' at one point each. Therefore we have the following diagram of simple moves.

$$\begin{tikzcd}
(\alpha_{n-1}, \alpha_n) \arrow[rr] & & (\alpha_{n-1}, \beta_n) \arrow[rr] & & (\beta_{n-1}, \beta_n) \arrow[dl] \\
& (\alpha_n, \gamma) \arrow[ul] & & (\beta_{n-1}, \gamma) \arrow[ll] \arrow[d] & \\
& (\alpha_n, \beta) \arrow[u] & & (\beta_{n-1}, \beta) \arrow[ll] \arrow[dr] & \\
(\alpha_n, \gamma') \arrow[uuu] \arrow[ur] & & & & (\beta_{n-1}, \gamma') \arrow[llll] \arrow[uuu]
\end{tikzcd}$$

The side quadrangles move only one circle of a cut system and they can be

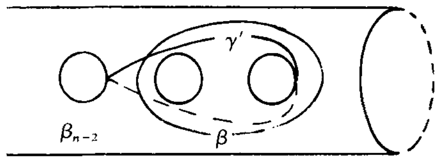

Fig. 4.

replaced by sums of triangles (of type L3). We have also two squares of type (L4). In view of (vi) and (vii) we can replace the small pentagon by the big pentagon in which γ is replaced by a fixed circle γ'. We can choose $h_1 = \tau_{n-1}$, $h_2 = h_3 = h_4 = \tau_{n-1} \cdot \alpha_n$ which satisfy the condition of relation (V). This gives the last relation

(viii) $\sigma h_1 \sigma h_2 \sigma h_3 \sigma h_4 \sigma \in H'$ where $h_1 = \tau_{n-1}$, $h_2 = h_3 = h_4 = \tau_{n-1} \cdot \alpha_n$.

The elements of H' corresponding to relations (iv)–(viii) can be explicitly computed.

The mapping class group G' admits a presentation with generators α_i, δ_{ij}, τ_i, ξ, σ and relations (i)–(viii).

4. Proof of Theorem 1

Let G denote a group with generators a_i, b_i, $i = 1, \cdots, n$ and d and with relations (A), (B), and (C). Let $\psi : G \to G'$ be an epimorphism defined by $\psi(a_i) = \alpha_i$, $\psi(b_i) = \beta_i$, $\psi(d) = \delta$. We want to prove that ψ is an isomorphism. We shall construct an inverse map $\phi : G' \to G$.

Define ϕ on the generators of G' as follows:

$$\phi(\xi) = x = b_n a_n a_n b_n,$$

$$\phi(\sigma) = s = a_n b_n a_n,$$

$$\phi(\alpha_i) = a_i \qquad \text{for } i = 1, \cdots, n,$$

$$\phi(\tau_i) = t_i = b_i a_i a_{i+1} b_i \qquad \text{for } i = 1, \cdots, n-1,$$

$$\phi(\delta_{ij}) = d_{ij} \qquad \text{where}$$

$$d_{ij} = (d)\bar{t}_2 \bar{t}_3 \cdots \bar{t}_{j-1} \bar{t}_1 \bar{t}_2 \cdots \bar{t}_{i-1} \qquad \text{if } j \leqq n,$$

$$d_{ij} = (d_{in})\bar{x}\bar{t}_{n-1}\bar{t}_{n-2} \cdots \bar{t}_{2n+1-j} \qquad \text{if } i < n < j < 2n+1-i,$$

$$d_{ij} = (d_{i-1,n})\bar{x}\bar{t}_{n-1}\bar{t}_{n-2} \cdots \bar{t}_{2n+1-j} \qquad \text{if } i \leqq n < 2n+1-i < j,$$

$$d_{ij} = (d_{n,n+2})\bar{x}\bar{t}_{n-2}\bar{t}_{n-3} \cdots \bar{t}_{2n+1-j}\bar{t}_{n-1} \cdots \bar{t}_{2n+1-i} \qquad \text{if } n < i,$$

$$d_{ij} = (s^4)d_{n-1,n}\bar{t}_{n-1}d_{n-2,n-1}\bar{t}_{n-2} \cdots d_{i,i+1}\bar{t}_i \qquad \text{if } i + j = 2n+1.$$

The map ϕ extends to a homomorphism if the relations (i)–(viii) are mapped by ϕ onto true relations in G. If it is true then we can check that $\psi\phi = \mathrm{Id}_{G'}$. Moreover ϕ is onto because $a_i = \phi(\alpha_i)$, $d = \phi(\delta_{1,2})$, $b_n = \bar{a}_n s \bar{a}_n$, and $b_i = (b_{i+1})t_i b_{i+1} a_{i+1}$ by (A), hence b_i belongs to $\phi(G')$ by induction downwards.

Therefore in order to prove Theorem 1 it is enough to show that the relations (i)–(viii) are satisfied in G.

We shall establish many relations in G from which the relations (i)–(viii) follow. Relations in (A) which do not contain d define a standard braid group. Relations in (A) which contain d and six other generators define a generalized braid group ([4]). For these groups the word problem has a simple solution therefore proofs based on these relations will usually be omitted. We shall denote by H_0 the subgroup of G generated by a_i's and d_{ij}'s and we shall denote by H the subgroup of G generated by H_0, x, and t_i's.

(1) a_i commutes with a_j for all i and j, by (A).

(2) $(a_i)t_i = a_{i+1}$, $(a_{i+1})t_i = a_i$, $(a_i)x = a_i$ for all i,
$(a_k)t_i = a_k$ for $k \neq i$, $k \neq i+1$, by (A).

(3) $(a_k)d_{ij} = a_k$ for all i, j, k, by (A), (2), and definitions.

(4) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i = 1, \cdots, n-1$,
t_i commutes with t_j for $|i-j| > 1$, by (A).

(5) $x t_{n-1} x$ commutes with t_{n-1}, by (A).

Definition. $e = (d) b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2$.

(6) $(a_1 b_1 a_2)^4 = de = t_1^2 a_1^2 a_2^2$, by (A) and (B).

(7) e commutes with d, a_1, b_1, a_2, and with a_i, b_i for $i > 2$, by (A) and (6).

(8) $(b_2)e = (e)\bar{b}_2$, by (A).

(9) d commutes with $(b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2)^2$, hence $e = (d)\bar{b}_2 \bar{a}_2 \bar{b}_1 \bar{a}_1 \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2$.

Proof.

$$(d)(b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2)^2 = (e) b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2$$

$$= ((\bar{d}) b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2) \cdot (a_1 b_1 a_2)^4 = (\bar{e}) \cdot (a_1 b_1 a_2)^4 = d, \qquad \text{by (6) and (A).}$$

□

(10) $(d_{i,i+1})\bar{t}_{i+1}\bar{t}_i = (d_{i,i+1})t_{i+1}t_i = d_{i+1,i+2}$ for $i = 1, \cdots, n-2$.

PROOF. For $i=1$

$$(d)t_2t_1t_1t_2=(e)\bar{b}_2\bar{a}_2\bar{b}_1\bar{a}_1a_3b_2a_2b_1b_1a_1a_2b_1b_2a_2a_3b_2$$

$$=(e)a_3b_2a_2b_1a_1a_1b_1a_2b_2\bar{a}_3=d, \qquad \text{by (A), (7) and (9).}$$

Let $u=t_3t_4\cdots t_{i+1}t_2t_3\cdots t_it_1t_2\cdots t_{i-1}$. Then by (A) and (4)

$$(d)u=d_{i,i+1}, \quad (d_{2,3})u=d_{i+1,i+2}, \quad (t_1)u=t_i, \quad (t_2)u=t_{i+1}.$$

Hence the conjugation by u takes the above relation onto (10). □

(11) $e=d_{2n-1,2n}$.

PROOF. Let $u=t_2t_3\cdots t_{n-1}t_1t_2\cdots t_{n-2}$. By the definitions we have $(d_{2n-1,2n})u=(d_{n-1,n})\bar{x}\bar{t}_{n-1}\bar{x}$ and by (10) $(d_{n-1,n})=(d)u$.

By (A) and (4) we have $(a_n)\bar{u}=a_2$, $(b_{n-1})\bar{u}=b_1$, $(a_{n-1})\bar{u}=a_1$,

$$(b_n)\bar{u}=(b_n)\bar{a}_n\bar{b}_{n-1}\bar{a}_{n-1}\cdots\bar{a}_3\bar{b}_2=(b_2)a_3b_3\cdots b_n.$$

Moreover

$$xt_{n-1}x=b_na_nb_{n-1}a_{n-1}a_{n-1}b_{n-1}a_nb_nt_{n-1}$$

hence

$$u\bar{x}\bar{t}_{n-1}\bar{x}\bar{u}=\bar{t}_1\bar{b}_n\bar{a}_n\cdots\bar{b}_2\bar{a}_2\bar{b}_1\bar{a}_1\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2a_3b_3\cdots a_nb_n.$$

Therefore

$$d_{2n-1,2n}=(d)\bar{b}_2\bar{a}_2\bar{b}_1\bar{a}_1\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2a_3b_3\cdots a_nb_n=e, \qquad \text{by (7), (9).}$$ □

DEFINITION. $e_{ij}=d_{2n+1-j,2n+1-i}$.

We introduce the above notation in order to simplify indices. There are some obvious relations between e_{ij}'s and t_k's. In particular, by an argument similar to the proof of (10), we can show the following:

(12) $(e_{i-1,i})t_it_{i-1}=(e_{i-1,i})\bar{t}_i\bar{t}_{i-1}=e_{i,i+1}$ for $i=2,\cdots,n-1$.

(13) $t_i^2=d_{i,i+1}e_{i,i+1}\bar{a}_i^2\bar{a}_{i+1}^2$ for $i=1,\cdots,n-1$.

PROOF. The relation is true for $i=1$, by (6).

Let $u=t_2t_3\cdots t_it_1t_2\cdots t_{i-1}$. Then, by (4), (10) and (12),

$$(d)u=d_{i,i+1}, \quad (a_1)u=a_i, \quad (a_2)u=a_{i+1}, \quad (e)u=e_{i,i+1}, \quad (t_1)u=t_i,$$

and relation (13) follows from (6). □

(14) For $i<n$, if $|k-i|=1$ then $(b_k)d_{i,i+1}=(d_{i,i+1})\bar{b}_k$, and if $|k-i|\neq 1$ then b_k commutes with $d_{i,i+1}$. The relations are also true if we replace $d_{i,i+1}$ by $e_{i,i+1}$.

PROOF. The relations follow from (A), (7), (8), by induction on i.

(15) b_i commutes with $d_{i-1,i}t_id_{i-1,i}$, hence t_i too.
b_{i-2} commutes with $d_{i-1,i}t_{i-2}d_{i-1,i}$, hence t_{i-2} too.
b_n commutes with $d_{n-1,n}xd_{n-1,n}$, hence x too.
The relations are also true if we replace $d_{i-1,i}$ by $e_{i-1,i}$.

PROOF. The relations follow from (A) and (14).

(16) b_n commutes with $d_{n-1,n+2}$.

PROOF.

$$d_{n-1,n+2}=\bar{d}_{n-1,n}t_{n-1}s^4\bar{t}_{n-1}d_{n-1,n}$$

$$=a^4_{n-1}\bar{d}_{n-1,n}\bar{b}_n\bar{a}_nb_{n-1}a^2_{n-1}b^2_{n-1}a^2_{n-1}b_{n-1}a_nb_nd_{n-1,n} \qquad \text{by (A) and (3).}$$

Relation (16) follows from (A) and (14). □

DEFINITION. We shall say that t_i *permutes* circles γ_i and γ_{i+1} *counterclockwise* and *permutes* circles γ_{2n-i} and γ_{2n+1-i} *counterclockwise*, $\bar{d}_{i,i+1}t_i$ *permutes* γ_i and γ_{i+1} *clockwise* and permutes γ_{2n-i} and γ_{2n+1-i} *counterclockwise*, x *permutes* γ_n and γ_{n+1} *counterclockwise*. The inverses *permute* in the *opposite directions*.

DEFINITION. A conjugation of d_{ij} by x, t_k, or $d_{k,k+1}\bar{t}_k$ is *proper* if it takes γ_i onto γ_{i+1} clockwise or onto γ_{i-1} counterclockwise or does not move it and it takes γ_j onto γ_{j+1} clockwise or onto γ_{j-1} counterclockwise or does not move it.

(17) A proper conjugation by t_k or by x takes d_{ij} onto some d_{pq}.

PROOF. If $i+j\neq 2n+1$ then (17) is true by (4), (5), and the definitions. Assume $i+j=2n+1$. Then b_n commutes with the factors of d_{ij}, by (A), (14) and (16), hence x also does. It remains to consider a conjugation by t_k. If $k=i$ or $k=i-1$ then the conjugation is not proper. If $k<i-1$ then obviously t_k commutes with d_{ij}. We have to prove that the same is true for $k>i$ and by (14) we may assume that $i=k-1$.

$$d_{k-1,j}=(d_{k+1,j-2})\bar{t}_kd_{k,k+1}\bar{t}_{k-1}d_{k-1,k}=(d_{k+1,j-2})\bar{t}_k\bar{t}_{k-1}\bar{t}_kd_{k-1,k}t_kd_{k-1,k},$$

by (10). Now b_k commutes with $d_{k-1,j}$ by (15), (14) and (A). □

(18) d_{ij} commutes with $d_{k,k+1}$ if all indices are distinct.

PROOF. If $k = n$ then $d_{k,k+1} = a_n^4 x^2$ and (18) follows from (17). If $k \neq n$ we have to consider two cases.

Case 1: $i + j \neq 2n + 1$. We can conjugate both elements by consecutive products of a form $t_r t_{r+1}$ or $\bar{t}_r \bar{t}_{r+1}$. For a suitable choice of positive and negative powers the conjugation is proper for d_{ij} and eventually $d_{k,k+1}$ becomes d or e. Further proper conjugation by a suitable product of x's and t_p's, $p \neq 2$, leads to one of the following pairs:

(a): (d, e),

(b): $(d, e_{1,3})$ (which is equivalent by conjugation to $(e, d_{1,3})$ and $(e, d_{1,4})$),

(c): $(e, d_{3,4})$ (equivalent to $(d, e_{3,4})$ by conjugation by $t_2 t_1 t_3 t_2$).

The first pair commutes by (7). The second pair commutes by (A) and the definitions. The last case requires a use of relation (C). Conjugate relation (C) by $\bar{t}_3 \bar{t}_2$. Then the left hand side becomes $d_{1,3} d_{1,4} d_{3,4} \bar{a}_1 \bar{a}_3 \bar{a}_4$. The right hand side becomes $(v) u b_1 \bar{a}_4 \bar{b}_3 \bar{a}_3 \bar{b}_2$. Now $(v) \bar{a}_4 \bar{b}_3 \bar{a}_3 \bar{b}_2 = (d) \bar{t}_2 \bar{b}_1 \bar{a}_1 = (d_{1,3}) \bar{b}_1 \bar{a}_1$ commutes with e by (b). $(u) \bar{a}_4 \bar{b}_3 \bar{a}_3 \bar{b}_2 = (d) \bar{t}_2 \bar{t}_3 \bar{a}_3 \bar{b}_2 = (d) \bar{t}_2 \bar{t}_3 = d_{1,4}$ commutes with e by (b). $(b_1) \bar{a}_4 \bar{b}_3 \bar{a}_3 \bar{b}_2 = b_1$. Therefore all factors in the new relation commute with e, with the exception of $d_{3,4}$. Hence $d_{3,4}$ also commutes with e.

Case 2: $i + j = 2n + 1$. If $k < i - 1$ or $k > j$ then, by Case 1, $d_{k,k+1}$ commutes with $d_{p,p+1}$ for $i \leqq p < n$. Therefore we may assume $i = n$ and $d_{k,k+1}$ commutes with $d_{n,n+1}$ by (14). If $i < k < j - 1$ we can conjugate $d_{k,k+1}$ properly by a suitable product of x's and t_p's, $p > i$, and we arrive at $k = j - 2$. Now conjugate by $w = \bar{d}_{i,i+1} t_i t_{i+1} \bar{d}_{i+1,i+2}$. $(d_{ij}) w = d_{i+2,j-2}$. $d_{k,k+1}$ commutes with $d_{i,i+1}$ by Case 1. $(d_{k,k+1}) t_i t_{i+1} = d_{j-1,j}$ commutes with $d_{i+1,i+2}$ by Case 1, and commutes with $d_{i+2,j-2}$ by the first part of Case 2. □

(19) A proper conjugation by $d_{k,k+1} \bar{t}_k$ takes d_{ij} onto some d_{pq}.

PROOF. If $i + j = 2n + 1$ then either we get $d_{i+1,j-1}$ or $d_{i-1,j+1}$, by the definitions, or d_{ij} commutes with $d_{k,k+1}$ and t_k, by (17) and (18). If $i + j \neq 2n + 1$ then either d_{ij} commutes with $d_{k,k+1}$ and t_k by (17) and (18), or conjugation by $\bar{t}_k$ is proper and d_{ij} commutes with $d_{k,k+1}$, or conjugation by t_k is proper and d_{ij} commutes with $e_{k,k+1}$, while $\bar{t}_k d_{k,k+1} = t_k \bar{e}_{k,k+1} a_k^2 a_{k+1}^2$, by (13). □

COROLLARY 1. *Consider a map f of a set of k distinct d_{ij}'s onto another set of k distinct d_{ij}'s. It also defines a map of a set of 2k indices onto another set of 2k indices. The map f can be realized by a product of proper conjugations if it preserves the order of indices and two indices sum up to $2n + 1$ if and only if their images do.*

PROOF. Every proper conjugation gives a map as in the corollary, by (17) and (19). Different elementary maps are described in the definition after relation (16). Corollary 1 follows. □

(20) d_{ij} commutes with d_{kl} if $i<j<k<l$ or $i<k<l<j$.

PROOF. By Corollary 1 we may assume $l=k+1$ or $j=i+1$. Then (20) follows from (18). □

(21) $(d_{ij})\bar{d}_{jk}=(d_{ij})d_{ik}$ if $i<j<k$.

PROOF. By Corollary 1 we have to consider the following 4 cases depending on the indices which sum up to $2n+1$. (i,j,k) may be equal to:

Case 1: $(1,2,3)$, $3\leqq n$. $(d)\bar{d}_{2,3}=(d)\bar{t}_1\bar{t}_2\bar{d}t_2t_1=(d)\bar{t}_2\bar{d}t_2t_1=(d)t_2^2t_1$, by (15). $(d)d_{1,3}=(d)t_2d\bar{t}_2=(d)\bar{t}_2\bar{t}_2=(d)\bar{t}_2\bar{t}_1t_1\bar{t}_2=(d)t_2t_1t_1\bar{t}_2=(d)t_2^2t_1$, by (15), (10) (4).

Case 2: $(n-1,n,n+1)$. $(d_{n-1,n})\bar{d}_{n,n+1}=(d_{n-1,n})\bar{x}^2=(d_{n-1,n})xd_{n-1,n}\bar{x}=(d_{n-1,n})d_{n-1,n+1}$, by (15).

Case 3: $(n,n+1,n+2)$. $(d_{n,n+1})\bar{d}_{n+1,n+2}=a_n^4\cdot(x^2)\bar{e}_{n-1,n}=a_n^4\cdot(x^2)e_{n-1,n}x=a_n^4\cdot(x^2)\bar{x}e_{n-1,n}x=(d_{n,n+1})d_{n,n+2}$, by (15).

Case 4: $(n-1,n,n+2)$.

$$
\begin{aligned}
(d_{n-1,n})\bar{d}_{n,n+2}&=(d_{n-1,n})t_{n-1}x\bar{d}_{n-1,n}\bar{x}\bar{t}_{n-1}\\
&=(d_{n-1,n})d_{n-1,n}x\bar{d}_{n-1,n}\bar{x}\bar{t}_{n-1}\\
&=(d_{n-1,n})\bar{x}\bar{d}_{n-1,n}xd_{n-1,n}\bar{t}_{n-1}\\
&=(d_{n-1,n})x^2d_{n-1,n}\bar{t}_{n-1}\\
&=(d_{n-1,n})d_{n-1,n+2}, \qquad \text{by (15).}
\end{aligned}
$$
□

(22) $(d_{ik})d_{ij}d_{ik}=(d_{ik})\bar{d}_{jk}$ if $i<j<k$.

PROOF. By (21) $\bar{d}_{ik}\bar{d}_{ij}d_{ik}=d_{jk}\bar{d}_{ij}\bar{d}_{jk}$, hence (22) is equivalent to $d_{jk}\bar{d}_{ij}\bar{d}_{jk}d_{ij}d_{ik}=d_{jk}d_{ik}\bar{d}_{jk}$ or $(d_{jk})d_{ij}=(d_{jk})\bar{d}_{ik}$.

We have again 4 cases as in the proof of (21).

Case 1: $(d_{2,3})d=(d_{2,3})\bar{d}_{1,3}$. When we conjugate by $\bar{t}_1\bar{t}_2$ we get $(d)d_{1,3}=(d)\bar{d}_{2,3}$. This is Case 1 of (21).

Case 2: $(d_{n,n+1})d_{n-1,n}=a_n^4(x^2)x\bar{d}_{n-1,n}\bar{x}=(d_{n,n+1})\bar{d}_{n-1,n+1}$, by (15).

Case 3: $(d_{n+1,n+2})d_{n,n+1}=(e_{n-1,n})x^2=(e_{n-1,n})\bar{x}\bar{e}_{n-1,n}x=(d_{n+1,n+2})\bar{d}_{n,n+2}$, by (15).

Case 4: $(d_{n,n+2})d_{n-1,n} = (d_{n-1,n})\bar{x}\bar{t}_{n-1}d_{n-1,n} = (d_{n-1,n})\bar{x}\bar{d}_{n-1,n}\bar{x}\bar{x}d_{n-1,n}\bar{t}_{n-1} = (d_{n,n+2})\bar{d}_{n-1,n+2}$, by (15). □

(23) $(d_{ik})\bar{d}_{ij}\bar{d}_{il}d_{ij}d_{il} = (d_{ik})\bar{d}_{jl}$ if $i < j < k < l$.

Proof. By (22) $\bar{d}_{ij}\bar{d}_{il}d_{ij}d_{il} = d_{il}d_{jl}\bar{d}_{il}\bar{d}_{jl}$, hence (23) is equivalent to: d_{ik} commutes with $d_{il}d_{jl}\bar{d}_{il}$. By (21) this is equivalent to: d_{ik} commutes with $(d_{jl})d_{kl} = (d_{jl})\bar{d}_{jk} = u$. It is also equivalent to: d_{jl} commutes with $(d_{ik})\bar{d}_{kl} = v$. By Corollary 1 we have to consider 7 cases depending on the indices which sum up to $2n+1$. (i,j,k,l) may be equal to:

Case 1: $(1,2,3,4)$, $4 \leqq n$. Conjugate by t_2. $(d_{1,3})t_2 = d$.

$(u)t_2 = (d_{2,4})\bar{d}_{2,3}t_2 = (d_{2,3})\bar{t}_3\bar{d}_{2,3}t_2 = (d_{2,3})t_3t_2 = d_{3,4}$, by (15) and (10). $d_{3,4}$ commutes with d, by (20).

Case 2: $(n, n+1, n+2, n+3)$. Conjugate by $\bar{t}_{n-2}$. $(d_{n,n+2})\bar{t}_{n-2} = d_{n,n+3}$.

$$(u)\bar{t}_{n-2} = (d_{n+1,n+3})\bar{d}_{n+1,n+2}\bar{t}_{n-2} = d_{n+1,n+2},$$

by (15). $d_{n,n+3}$ commutes with $d_{n+1,n+2}$, by (20).

Case 3: $(n-1, n, n+2, n+3)$. Conjugate by t_{n-2}. $(d_{n,n+3})t_{n-2} = d_{n,n+2}$.

$$(v)t_{n-2} = (d_{n-1,n+2})\bar{d}_{n+2,n+3}t_{n-2} = (d_{n-1,n+2})d_{n-2,n-1}\bar{t}_{n-2} = d_{n-2,n+3},$$

by (13). $d_{n-2,n+3}$ commutes with $d_{n,n+2}$, by (20).

Case 4: $(n-2, n+1, n+2, n+3)$. Conjugate by t_{n-1}. $(d_{n-2,n+2})t_{n-1} = d_{n-2,n+1}$.

$$(u)t_{n-1} = (d_{n+1,n+3})d_{n+2,n+3}t_{n-1} = (e_{n-2,n-1})t_{n-1}e_{n-2,n-1}t_{n-1} = e_{n-2,n-1} = d_{n+2,n+3},$$

by (15). $d_{n-2,n+1}$ commutes with $d_{n+2,n+3}$, by (20).

Case 5: $(n-1, n, n+1, m)$. Conjugate by x. $(d_{n-1,n+1})x = d_{n-1,n}$.

$$(u)x = (d_{n,m})d_{n+1,m}x = (d_{n+1,m})xd_{n+1,m}x = d_{n+1,m}$$

because, by (A), (14) and the definitions, $(b_n)d_{n+1,m} = (d_{n+1,m})\bar{b}_n$. $d_{n+1,m}$ commutes with $d_{n-1,n}$, by (20).

Case 6: $(n-2, n-1, n, n+2)$. Conjugate by t_{n-1}. $(d_{n-2,n})t_{n-1} = d_{n-2,n-1}$.

$$(u)t_{n-1} = (d_{n-1,n+2})\bar{d}_{n-1,n}t_{n-1} = d_{n,n+1}.$$

$d_{n,n+1}$ commutes with $d_{n-2,n-1}$, by (20).

Case 7: $(n-2, n-1, n, n+1)$. Conjugate by $\bar{x}$. $(d_{n-2,n})\bar{x} = d_{n-2,n+1}$.

$$(u)\bar{x} = (d_{n-1,n+1})d_{n,n+1}\bar{x} = (d_{n-1,n+1})x = d_{n-1,n}.$$

$d_{n-1,n}$ commutes with $d_{n-2,n+1}$, by (20). □

We have already established that relations (i) are satisfied in G. Therefore the map ϕ described at the beginning of this section extends to an isomorphism of H_0' onto H_0. It follows that it is enough to prove relations (ii)–(iii) in their present form without an explicit knowledge of the corresponding elements of H_0'.

(24) $x^2 = d_{n,n+1}\bar{a}^4 \in H_0$.

(25) $(d_{ij})x \in H_0$, $(d_{ij})t_k \in H_0$, for all i, j, k.

PROOF. Either conjugation by x or by $\bar{x}$ is proper. Either conjugation by t_k or by $\bar{t}_k$ or by $t_k\bar{d}_{k,k+1}$ or by $\bar{t}_k d_{k,k+1}$ is proper. Relation (25) follows from (24), (13), (17), (19).

(26) $[(x)t_{n-1}t_{n-2}\cdots t_i, (x)t_{n-1}t_{n-2}\cdots t_j] \in H_0$ for $i<j$.

PROOF. By (2) and (25) we may conjugate the relation by

$$u = \bar{t}_j\bar{t}_{j+1}\cdots\bar{t}_{n-1}\bar{t}_i\bar{t}_{i+1}\cdots\bar{t}_{n-2}.$$

We get $\bar{t}_{n-1}xt_{n-1}x\bar{t}_{n-1}\bar{x}\bar{t}_{n-1}\bar{x} = \bar{t}^2_{n-1}xt^2_{n-1}\bar{x} \in H_0$, by (5) and (25).

(27) t_i commutes with $(x)t_{n-1}t_{n-2}\cdots t_j$, for $i \neq j$, $i \neq j-1$, by (A) and (4).

We have established by now that the relations (i)–(iii) are satisfied in G. Therefore the map ϕ extends to an isomorphism of H' onto H. It follows that it is enough to prove relations (iv)–(viii) in their present form without an explicit knowledge of the corresponding elements of H'.

(28) a_n and b_n commute with the following elements of $H: t_1, \cdots, t_{n-2}$, a_{n-1}, $d_{n-2,n+2}$, $(s^2)d_{n-1,n}\bar{t}_{n-1}$, $d_{n,n+1}d_{n,n+2}d_{n+1,n+2}\bar{a}_{n-1}\bar{a}^2_n$.

PROOF. $d_{n-2,n+2} = (d_{n-2,n-1})\bar{t}_{n-1}\bar{x}\bar{t}_{n-1}$ commutes with b_n, by (A) and (14). $(s^2)d_{n-1,n}\bar{t}_{n-1} = \bar{d}_{n-1,n}\bar{b}_n\bar{a}_n b_{n-1}a^2_{n-1}b_{n-1}a_n b_n d_{n-1,n}a^2_{n-1}$ commutes with b_n by (14).

$$d_{n,n+1}d_{n,n+2}d_{n+1,n+2}\bar{a}^2_n\bar{a}_{n-1} = x^2a^2_n\bar{x}d_{n+1,n+2}xd_{n+1n+2}\bar{a}_{n-1}$$

commutes with b_n, by (15) and (A). All other relations follow from (A) and (3). □

(29) $s^2 = a^2_n x \in H$.

(30) *shshs* $\in H$ for each of the following choices of $h: h = a_n$, $h = d_{n-1,n}$, $h = d_{n-1,n+2}d_{n,n+2}d_{n-1,n}\bar{a}^2_{n-1}\bar{a}_n$, $h = d_{n-2,n-1}d_{n-2,n}d_{n-1,n}\bar{a}_{n-2}\bar{a}_{n-1}\bar{a}_n$.

PROOF. It is enough to prove that $(b_n)h = (h)\bar{b}_n$ and $(a_n)h = a_n$, because then $shshs = a_n x h x a_n \in H$. The relations are true for $h = a_n$ and for $h = d_{n-1,n}$. Consider the third h. $d_{n-1,n+2}$ commutes with b_n, by (16). $d_{n,n+2} = (e_{n-1,n})x = (d_{n-1,n})\bar{x}\bar{t}_{n-1}$.

$$(b_n)h = (b_n)\bar{x}e_{n-1,n}xd_{n-1,n}\bar{a}_n = (e_{n-1,n})b_na_nd_{n-1,n}$$

$$= \bar{d}_{n-1,n}\bar{a}_n\bar{b}_n\bar{d}_{n-1,n}a_n^2a_{n-1}^2t_{n-1}^2b_na_nd_{n-1,n},$$

by (A), (14), (13).

$$(h)\bar{b}_n = b_nt_{n-1}\bar{d}_{n-1,n}a_n^4x^2d_{n-1,n}\bar{t}_{n-1}t_{n-1}xd_{n-1,n}\bar{x}\bar{t}_{n-1}d_{n-1,n}\bar{a}_{n-1}^2\bar{a}_n\bar{b}_n$$

$$= b_nt_{n-1}a_n^2xd_{n-1,n}xd_{n-1,n}\bar{t}_{n-1}\bar{a}_n\bar{b}_n,$$

by (15). The equality $(b_n)h = (h)\bar{b}_n$ follows from (A) and (14).

Consider now the last h. Let

$$w = t_{n-3}t_{n-4}\cdots t_1t_{n-2}t_{n-3}\cdots t_2t_{n-1}t_{n-2}\cdots t_3b_na_n\cdots b_4a_4.$$

Then $h_1 = (h)w = dd_{1,3}d_{2,3}\bar{a}_1\bar{a}_2\bar{a}_3$, $(b_n)w = b_3$ and we have to prove that $(b_3)h_1 = (h_1)\bar{b}_3$. By relation (C) $h_1 = (d)\bar{b}_2\bar{a}_2\bar{b}_1\bar{a}_1ub_1a_2b_2a_3b_3$, where $u = (d)\bar{t}_2\bar{b}_3\bar{a}_3$. It follows from (A) that $(b_3)h_1 = (h_1)\bar{b}_3$. □

(31) s commutes with $hs\bar{h}$, where $h = \bar{d}_{n-1,n}t_{n-1}a_n$.

PROOF. $hs\bar{h} = \bar{d}_{n-1,n}a_{n-1}^2b_{n-1}a_nb_n\bar{a}_n\bar{b}_{n-1}d_{n-1,n} = a_{n-1}^2\bar{d}_{n-1,n}\bar{b}_n\bar{a}_nb_{n-1}a_nb_nd_{n-1,n}$. Therefore a_n and b_n commute with $hs\bar{h}$, by (A) and (14). □

(32) $st_{n-1}s(t_{n-1}a_ns)^3 \in H$.

PROOF. $st_{n-1}s(t_{n-1}a_ns)^3 = a_{n-1}^3a_n^3x(t_{n-1}x)^3$, by (A). □

It follows from (28)–(32) that relations (iv)–(viii) are satisfied in G, hence ϕ can be extended to an isomorphism of the mapping class group $G' = M_{n,1}$ onto the group G. This concludes the proof of Theorem 1.

5. Proof of Theorem 2

We shall consider now a mapping class group $M = M_{n,0}$ of a closed surface $F_{n,0}$. We shall keep the notation from the previous sections. In particular G is the mapping class group of a surface $F_{n,1}$ with one boundary component. It follows from [2], theorem 4.3, and from [6], section 4, that there exist two exact sequences

$$1 \longrightarrow \pi_1(F_{n,0}, p) \xrightarrow{f_1} \Lambda_n \xrightarrow{f_2} M \longrightarrow 1$$

and

$$1 \longrightarrow Z \xrightarrow{f_3} G \xrightarrow{f_4} \Lambda_n \longrightarrow 1.$$

Here Λ_n is the group of isotopy classes of orientation preserving diffeomorphisms of $F_{n,0}$ which fix a base point p, and f_4 is defined by capping $\partial F_{n,1}$ with a disc, which contains p, and extending each map over the disc by the identity.

We have to find the kernel of the composition $f_2 f_4$. The kernel of f_4 is generated by the Dehn twist ω with respect to the boundary $\partial F_{n,1}$. It follows from [2] that the kernel of $f_2 f_4$ is generated by ω and by "spin" maps $\delta\bar{\delta}'$ where δ and δ' are nonseparating simple closed curves separated only by the "hole" bounded by $\partial F_{n,1}$. Clearly all spin maps are conjugate in the group G. Let us choose a pair δ_n, ε_n on Fig. 5. Then M admits a presentation with relations (A), (B), (C) and relations $\delta_n = \varepsilon_n$, $\omega = 1$. Moreover δ_n represents in G the element d_n from relation (D) and $\varepsilon_n = (\delta_n) b_n a_n \cdots b_1 a_1 a_1 b_1 \cdots a_n b_n$ in G.

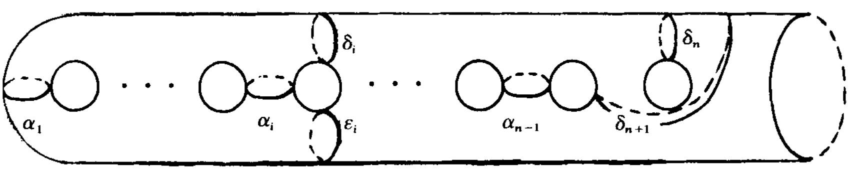

Fig. 5.

Let M' denote the quotient of G by the relation $\delta_n = \varepsilon_n$. In order to prove Theorem 2 it suffices to show that $\omega = 1$ in M'.

We observe that in G, $\omega = (a_1 b_1 \cdots a_n b_n d_n)^{2n+2}$, $(d_n) b_n = (b_n)\bar{d}_n$, and d_n commutes with all other b_i's and a_j's. Also

$$(a_1 b_1 \cdots a_n)^{2n} = \delta_n \varepsilon_n, \quad (a_1 b_1 \cdots a_{n-1})^{2n-2} = \delta_{n-1}\varepsilon_{n-1} \quad \text{and} \quad (a_n b_n d_n)^4 = \delta_{n-1}\delta_{n+1}.$$

The map $\delta_{n+1}\bar{\varepsilon}_{n-1}$ is a spin map, hence trivial in M'. Therefore in M'

$$(a_1 b_1 \cdots a_n)^{2n} = d_n^2 \quad \text{and} \quad (a_1 b_1 \cdots a_{n-1})^{2n-2} = (a_n b_n d_n)^4.$$

We shall abbreviate the product $b_i a_i b_{i-1} \cdots b_1 a_1 a_1 b_1 \cdots b_{i-1} a_i b_i$ by $b_i \cdots b_i$. Then by relations (A) and by the new relations in M' we have

$$\omega = (a_1 b_1 \cdots b_n d_n)^{2n+2} = (a_1 b_1 \cdots a_n)^{2n} \cdot (b_n \cdots b_n) \cdot (d_n b_n \cdots b_n d_n),$$

$$(a_1 b_1 \cdots a_n)^{2n} = (a_1 b_1 \cdots a_{n-1})^{2n-2} \cdot (b_{n-1} \cdots b_{n-1}) \cdot (a_n b_{n-1} \cdots b_{n-1} a_n).$$

Therefore

$$a_n b_{n-1} \cdots b_{n-1} a_n = d_n^2 (b_{n-1} \cdots b_{n-1})^{-1} \cdot (d_n b_n a_n)^{-4}$$

and

$$\omega = d_n^2 b_n d_n^2 (b_{n-1} \cdots b_{n-1})^{-1} \cdot \bar{a}_n \bar{b}_n \bar{d}_n (\bar{a}_n \bar{b}_n \bar{d}_n)^3 \cdot b_n d_n b_n \cdots b_n d_n .$$

Since in M', $d_n b_n \cdots b_n d_n$ commutes with d_n and with all a_i's and b_i's we have $\omega = d_n^2 b_n d_n^2 a_n b_n d_n (\bar{a}_n \bar{b}_n \bar{d}_n)^3 \cdot b_n = 1$.

This concludes the proof of Theorem 2.

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MATHEMATICS FACULTY
TECHNION — ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA, ISRAEL